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FORMULAE FOR ASKEY-WILSON MOMENTS AND ENUMERATION OF STAIRCASE TABLEAUX

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Abstract. We explain how the moments of the (weight function of the) Askey Wilson polynomials are related to the enumeration of the *staircase tableaux* introduced by the first and fourth authors [11, 12]. This gives us a direct combinatorial formula for these moments, which is related to, but more elegant than the formula given in [11]. Then we use techniques developed by Ismail and the third author to give explicit formulae for these moments and for the enumeration of staircase tableaux. Finally we study the enumeration of staircase tableaux at various specializations of the parameterizations; for example, we obtain the Catalan numbers, Fibonacci numbers, Eulerian numbers, the number of permutations, and the number of matchings.

[Keywords: staircase tableaux, asymmetric exclusion process, Askey-Wilson polynomials, permutations, matchings]

Contents

1. Introduction

In recent work [11, 12] the first and fourth authors presented a new combinatorial object that they called *staircase tableaux*. They used these objects to solve two related problems: to give a combinatorial formula for the stationary distribution of the asymmetric exclusion process on a one-dimensional lattice with open boundaries, where all parameters $\alpha, \beta, \gamma, \delta, q$ are general; and to give a combinatorial formula for the moments of the (weight function of the) Askey-Wilson polynomials. In this paper we build upon that work, and give a somewhat simpler combinatorial formula for the Askey-Wilson moments. We also use work of Ismail and the third author to give an explicit formula for the Askey-Wilson moments. Finally we study some special cases and explore the combinatorial properties of staircase tableaux: for example, we highlight a forest structure underlying staircase tableaux.

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Definition 1.1. *A* staircase tableau *of size n is a Young diagram of "staircase" shape $(n, n-1, \dots, 2, 1)$ such that boxes are either empty or labeled with α, β, γ, or δ, subject to the following conditions:*

- *no box along the diagonal is empty;*
- *all boxes in the same row and to the left of a β or a δ are empty;*
- *all boxes in the same column and above an α or a γ are empty.*

The type type$(\mathcal{T})$ *of a staircase tableau $\mathcal{T}$ is a word in $\{\circ, \bullet\}^n$ obtained by reading the diagonal boxes from northeast to southwest and writing a $\bullet$ for each α or δ, and a $\circ$ for each β or γ.*

See the left of Figure 1 for an example.

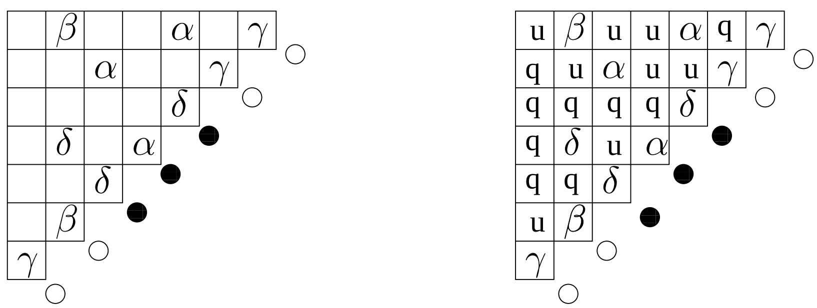

FIGURE 1. A staircase tableau of size 7 and type $\circ \circ \bullet \bullet \bullet \circ \circ$

Staircase tableaux with no γ's or δ's are in bijection with permutation tableaux [26, 28] and alternative tableaux [31]. See [12] for more details.

Definition 1.2. *The* weight wt$(\mathcal{T})$ *of a staircase tableau $\mathcal{T}$ is a monomial in $\alpha, \beta, \gamma, \delta, q$, and u, which we obtain as follows. Every blank box of $\mathcal{T}$ is assigned a q or u, based on the label of the closest labeled box to its right in the same row and the label of the closest labeled box below it in the same column, such that:*

- *every blank box which sees a β to its right gets a u;*
- *every blank box which sees a δ to its right gets a q;*
- *every blank box which sees an α or γ to its right, and an α or δ below it, gets a u;*
- *every blank box which sees an α or γ to its right, and a β or γ below it, gets a q.*

After filling all blank boxes, we define wt$(\mathcal{T})$ *to be the product of all labels in all boxes.*

The right of Figure 1 shows that the weight of the staircase tableau is $\alpha^3\beta^2\gamma^3\delta^3q^9u^8$.

Remark 1.3. *The weight of a staircase tableau always has degree $n(n+1)/2$. For convenience, we will usually set $u = 1$, since this results in no loss of information.*

We define

$$Z_n(\alpha, \beta, \gamma, \delta; q, u) = \sum_{\mathcal{T} \text{ of size } n} \mathrm{wt}(\mathcal{T}).$$

This is the generating polynomial for staircase tableaux of size n. We also use the symbol $Z_n(\alpha, \beta, \gamma, \delta; q)$ to denote the same quantity with $u = 1$.

We now review the definition of the (partially) asymmetric exclusion process [14], a classical model in statistical mechanics. This is a model of particles hopping on a lattice with n sites, where particles may hop to adjacent sites in the lattice, and may enter and

exit the lattice at both the left and right boundaries, subject to the condition that at most one particle may occupy a given site. The model can be described by a discrete-time Markov chain [14, 15] as follows.

Definition 1.4. *Let α, β, γ, δ, q, and u be constants such that $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, $0 \leq \gamma \leq 1$, $0 \leq \delta \leq 1$, $0 \leq q \leq 1$, and $0 \leq u \leq 1$. The ASEP is the Markov chain on the 2^n words in the language language $\{\circ, \bullet\}^*$, with transition probabilities:*

- *If $X = A\bullet\circ B$ and $Y = A\circ\bullet B$ then $P_{X,Y} = \frac{u}{n+1}$ (particle hops right) and $P_{Y,X} = \frac{q}{n+1}$ (particle hops left).*
- *If $X = \circ B$ and $Y = \bullet B$ then $P_{X,Y} = \frac{\alpha}{n+1}$*
- *If $X = B\bullet$ and $Y = B\circ$ then $P_{X,Y} = \frac{\beta}{n+1}$*
- *If $X = \bullet B$ and $Y = \circ B$ then $P_{X,Y} = \frac{\gamma}{n+1}$*
- *If $X = B\circ$ and $Y = B\bullet$ then $P_{X,Y} = \frac{\delta}{n+1}$*
- *Otherwise $P_{X,Y} = 0$ for $Y \neq X$ and $P_{X,X} = 1 - \sum_{X \neq Y} P_{X,Y}$.*

In the long time limit, the system reaches a steady state where all the probabilities $P_n(\sigma_1, \sigma_2, \ldots, \sigma_n)$ of finding the system in configuration $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ are stationary. Let

$$Z_\sigma(\alpha, \beta, \gamma, \delta; q, u) = \sum_{\mathcal{T} \text{ of type } \sigma} \mathrm{wt}(\mathcal{T}). \tag{1.1}$$

This is just the generating polynomial for staircase tableaux of a given type. As before, we lose no information by setting $u = 1$, and in that case let $Z_\sigma(\alpha, \beta, \gamma, \delta; q) := Z_\sigma(\alpha, \beta, \gamma, \delta; q, 1)$.

Theorem 1.5. [11, Corteel, Williams] *Consider any state σ of the ASEP with n sites, where the parameters $\alpha, \beta, \gamma, \delta, q$ and u are general. Then the steady state probability that the ASEP is at state σ is precisely*

$$\frac{Z_\sigma(\alpha, \beta, \gamma, \delta; q, u)}{Z_n(\alpha, \beta, \gamma, \delta; q, u)}.$$

By Theorem 1.5, we can call $Z_n(\alpha, \beta, \gamma, \delta; q, u)$ the *partition function* of the ASEP.

We now review the definition of the Askey-Wilson polynomials; these are orthogonal polynomials with five free parameters a, b, c, d, q, which reside at the top of the hierarchy of the one-variable q-orthogonal polynomials in the Askey scheme [2, 18].

Remark 1.6. *When working with Askey-Wilson polynomials, it will be convenient to use three variables x, θ, z, which are related to each other as follows:*

$$x = \cos\theta, \;\; z = e^{i\theta}, \;\; x = \frac{z + z^{-1}}{2}$$

Definition 1.7. *The Askey-Wilson polynomial $P_n(x) = P_n(x; a, b, c, d|q)$ is explicitly defined to be*

$$a^{-n}(ab, ac, ad; q)_n \sum_{k=0}^{n} \frac{(q^{-n}, q^{n-1}abcd, ae^{i\theta}, ae^{-i\theta}; q)_k}{(ab, ac, ad, q; q)_k} q^k,$$

where n is a non-negative integer and

$$(a_1, a_2, \cdots, a_s; q)_n = \prod_{r=1}^{s} \prod_{k=0}^{n-1} (1 - a_r q^k).$$

For $|a|, |b|, |c|, |d| < 1$, the orthogonality is expressed by

$$\oint_C \frac{dz}{4\pi i z} w\left(\frac{z+z^{-1}}{2}\right) P_m\left(\frac{z+z^{-1}}{2}\right) P_n\left(\frac{z+z^{-1}}{2}\right) = \frac{h_n}{h_0}\delta_{mn},$$

where the integral contour C is a closed path which encloses the poles at $z = aq^k$, bq^k, cq^k, dq^k $(k \in \mathbb{Z}_+)$ and excludes the poles at $z = (aq^k)^{-1}$, $(bq^k)^{-1}$, $(cq^k)^{-1}$, $(dq^k)^{-1}$ $(k \in \mathbb{Z}_+)$, and where

$$h_0 = h_0(a,b,c,d,q) = \frac{(abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd;q)_\infty},$$

$$\frac{h_n}{h_0} = \frac{(1-q^{n-1}abcd)(q,ab,ac,ad,bc,bd,cd;q)_n}{(1-q^{2n-1}abcd)(abcd;q)_n},$$

$$w(\cos\theta) = \frac{(e^{2i\theta}, e^{-2i\theta};q)_\infty}{h_0(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}, de^{i\theta}, de^{-i\theta};q)_\infty}.$$

(In the other parameter region, the orthogonality is continued analytically.)

Remark 1.8. *We remark that our definition of the weight function above differs slightly from the definition given in* [2]*; the weight function in* [2] *did not have the* h_0 *in the denominator. Our convention simplifies some of the formulas to come.*

Definition 1.9. *The moments of the (weight function of the) Askey-Wilson polynomials – which we sometimes refer to as simply the* Askey-Wilson moments *– are defined by*

$$\mu_n(a,b,c,d|q) = \oint_C \frac{dz}{4\pi i z} w\left(\frac{z+z^{-1}}{2}\right)\left(\frac{z+z^{-1}}{2}\right)^k.$$

The combinatorial formula given in [11, 12] is the following.

Theorem 1.10. [11, Corteel, Williams] *The nth Askey-Wilson moment is given by*

$$\mu_n(a,b,c,d|q) = \sum_{\ell=0}^{n} (-1)^{n-\ell}\binom{n}{\ell}\left(\frac{1-q}{2}\right)^\ell \frac{Z_\ell(\alpha,\beta,\gamma,\delta;q)}{\prod_{j=0}^{\ell-1}(\alpha\beta-\gamma\delta q^j)},$$

where

$$\alpha = \frac{1-q}{1+ac+a+c},\ \beta = \frac{1-q}{1+bd+b+d},\ \gamma = \frac{-(1-q)ac}{1+ac+a+c},\ \delta = \frac{-(1-q)bd}{1+bd+b+d}. \tag{1.2}$$

Note that this formula is not totally satisfactory as it has an alternating sum.

In the first half of this paper we give combinatorial and explicit formulas for Askey-Wilson polynomials and generating functions of staircase tableaux, including a combinatorial formula for the moments "on the nose".

To give this formula, we define $t(\mathcal{T})$ to be the number of (black) particles in type($\mathcal{T}$). For example the tableau $\mathcal{T}$ in Figure 1 has $t(\mathcal{T}) = 3$. We define the *fugacity partition function* of the ASEP to be

$$Z_n(y;\alpha,\beta,\gamma,\delta;q) = \sum_{\mathcal{T}\text{ of size } n} \mathrm{wt}(\mathcal{T}) y^{t(\mathcal{T})},$$

because this formula is a y-analogue of the partition function.

The exponent of y keeps track of the number of black particles in each state.

Theorem 1.11. *The n^{th} Askey-Wilson moment is equal to*

$$\mu_n(a,b,c,d|q) = \frac{(1-q)^n}{2^n i^n \prod_{j=0}^{n-1}(\alpha\beta - \gamma\delta q^j)} Z_n(-1;\alpha,\beta,\gamma,\delta;q),$$

where $i^2 = -1$ and

(1.3)

$$\alpha = \frac{1-q}{1-ac+ai+ci}, \quad \beta = \frac{1-q}{1-bd-bi-di}, \quad \gamma = \frac{(1-q)ac}{1-ac+ai+ci}, \quad \delta = \frac{(1-q)bd}{1-bd-bi-di}.$$

Note that the Askey-Wilson moments are in general rational expressions (with a simple denominator); the coefficients are not all positive, but they are all real. See Example 3.4. However, it's not at all clear from Theorem 1.11 that the coefficients are real.

Using work of Ismail and the third author [19], we also give explicit formulas for both the Askey-Wilson moments and the fugacity partition function of the ASEP.

Theorem 1.12. *The moments $\mu_n(a,b,c,d|q)$ are*

$$\frac{1}{2^n}\sum_{k=0}^{n} \frac{(ab,ac,ad;q)_k}{(abcd;q)_k} q^k \sum_{j=0}^{k} q^{-j^2} a^{-2j} \frac{(q^j a + q^{-j}/a)^n}{(q,q^{-2j+1}/a^2;q)_j (q,q^{2j+1}a^2;q)_{k-j}}.$$

Theorem 1.13. *The fugacity partition function $Z_n(y;\alpha,\beta,\gamma,\delta;q)$ of the ASEP is*

$$\begin{aligned} Z_n(y;\alpha,\beta,\gamma,\delta;q) =& (abcd;q)_n \left(\frac{\alpha\beta}{1-q}\right)^n \sum_{k=0}^{n} \frac{(ab,ac/y,ad;q)_k}{(abcd;q)_k} q^k \\ & \times \sum_{j=0}^{k} q^{-j^2}(a^2/y)^{-j} \frac{(1+y+q^j a + q^{-j}y/a)^n}{(q,q^{-2j+1}y/a^2;q)_j (q,a^2q^{1+2j}/y;q)_{k-j}}, \end{aligned}$$

where

$$a = \frac{1-q-\alpha+\gamma+\sqrt{(1-q-\alpha+\gamma)^2+4\alpha\gamma}}{2\alpha}, \; b = \frac{1-q-\beta+\delta+\sqrt{(1-q-\beta+\delta)^2+4\beta\delta}}{2\beta},$$

(1.4)

$$c = \frac{1-q-\alpha+\gamma-\sqrt{(1-q-\alpha+\gamma)^2+4\alpha\gamma}}{2\alpha}, \; d = \frac{1-q-\beta+\delta-\sqrt{(1-q-\beta+\delta)^2+4\beta\delta}}{2\beta}.$$

(Note that these expressions for a,b,c,d invert the transformation given in Theorem 1.10.)

In the second half of this paper we explore the wonderful combinatorial properties of staircase tableaux. For example, when we specialize some of the variables in the generating polynomial $Z_n(y;\alpha,\beta,\gamma,\delta;q)$ for staircase tableaux, we get some nice formulas and combinatorial numbers; see Table 1 below. The reference for each statement in the table is given in the rightmost column. A few of the simple statements we leave as exercises.

The paper is organized as follows. In Section 2, we explain how the Askey-Wilson moments are related to the generating polynomial of staircase tableaux. In Section 3, we compute explicit formulas for the moments and for the generating polynomials of staircase tableaux. In Section 4, we study some specializations of the generating polynomials, namely $q=0$, $q=1$ and $\delta=0$. In those cases we highlight the connection to other combinatorial objects. We conclude this paper with a list of open problems.

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α	β	γ	δ	q	y	$Z_n(y;\alpha,\beta,\gamma,\delta;q)$	Reference
α	β	γ	δ	1	1	$\prod_{j=0}^{n-1}(\alpha+\beta+\gamma+\delta+j(\alpha+\gamma)(\beta+\delta))$	Theorem 4.1
α	β	γ	$-\beta$	q	1	$\prod_{j=0}^{n-1}(\alpha+q^j\gamma)$	Proposition 4.9
α	β	γ	β	q	-1	$(-1)^n\prod_{j=0}^{n-1}(\alpha-q^j\gamma)$	Proposition 4.10
α	0	γ	0	q	y	$\prod_{j=0}^{n-1}(y\alpha+q^j\gamma)$	Exercise.
0	β	γ	0	q	y	$\prod_{j=0}^{n-1}(\beta+\beta\gamma[j]_q+\gamma q^j)$	See [13].
α	β	0	0	q	y	[20, Theorem 1.3.1]	See [20].
1	1	0	0	q	y	$\sum_{k=1}^{n+1}E_{k,n+1}(q)y^{k-1}$	See [32, Section 5] for definition of $E_{k,n}(q)$; use [6, Theorem 8].
1	1	0	0	-1	y	$(y+1)^n$	[32, Proposition 5.7]
α	α	α	α	-1	y	0 for $n\geq 3$	See Proposition 4.11.
1	1	1	1	1	y	$2^n(y+1)^n n!$	Exercise.
1	1	1	1	1	1	$4^n n! = 4n!!!!$	Follows from Theorem 4.1.
1	1	1	0	1	1	$(2n+1)!!$	Follows from Theorem 4.1.
1	1	0	0	1	1	$(n+1)!$	Follows from Theorem 4.1.
1	1	0	0	0	1	$C_{n+1}=\frac{1}{n+2}\binom{2n+2}{n+1}$	Follows from [32, Section 5].
1	1	1	0	0	0	F_{2n+1} (Fibonacci)	See Corollary 3.10.
1	1	1	0	0	1	Sloane A026671	See Corollary 3.10.
1	1	1	1	0	0	$2F_{2n}$ (Fibonacci)	See Corollary 3.12.

TABLE 1.

2. A combinatorial formula for Askey-Wilson moments

The goal of this section is to prove Theorem 1.11. Before we do so, we review the connection between orthogonal polynomials and tridiagonal matrices. Recall that by Favard's Theorem, orthogonal polynomials satisfy a three-term recurrence.

Theorem 2.1. *Let $\{P_k(x)\}_{k\geq 0}$ be a family of monic orthogonal polynomials. Then there exist coefficients $\{b_k\}_{k\geq 0}$ and $\{\lambda_k\}_{k\geq 1}$ such that $P_{k+1}(x)=(x-b_k)P_k(x)-\lambda_k P_{k-1}(x)$.*

By work of [16, 30], the nth moment of a family of monic orthogonal polynomials can be computed using a tridiagonal matrix, whose rows contain the information of the three-term recurrence. In what follows, $\langle\widetilde{W}|=(1,0,0,\dots)$ and $|\widetilde{V}\rangle=\langle\widetilde{W}|^T$. Note that we use the bra and ket notations to indicate row and column vectors, respectively.

Theorem 2.2. [16, 30] *Consider a family of monic orthogonal polynomials $\{P_k(x)\}_{k\geq 0}$ which satisfy the three-term recurrence $P_{k+1}(x)=(x-b_k)P_k(x)-\lambda_k P_{k-1}(x)$, for $\{b_k\}_{k\geq 0}$ and $\{\lambda_k\}_{k\geq 1}$. Then the nth moment μ_n of $\{P_k(x)\}_{k\geq 0}$ is equal to $\langle\widetilde{W}|M^n|\widetilde{V}\rangle$, where $M=$*

$(m_{ij})_{i,j\geq 0}$ is the tridiagonal matrix with rows and columns indexed by the non-negative integers, such that $m_{i,i-1}=\lambda_i$, $m_{ii}=b_i$, and $m_{i,i+1}=1$.

See [7] for a simple proof of Theorem 2.2. We also note the following.

Remark 2.3. *The polynomials defined by $Q_{k+1}(x)=(x-b_k)Q_k(x)-\lambda_k Q_{k-1}(x)$ have the same moments as the polynomials defined by $a_k Q'_{k+1}(x)=(x-b_k)Q'_k(x)-c_k Q'_{k-1}(x)$ as long as $a_{k-1}c_k=\lambda_k$.*

Now consider the following tridiagonal matrices, which were introduced by Uchiyama, Sasamoto and Wadati in [29].

$$\mathsf{d}=\begin{bmatrix} d_0^\natural & d_0^\sharp & 0 & \cdots \\ d_0^\flat & d_1^\natural & d_1^\sharp & \\ 0 & d_1^\flat & d_2^\natural & \ddots \\ \vdots & & \ddots & \ddots \end{bmatrix}, \qquad \mathsf{e}=\begin{bmatrix} e_0^\natural & e_0^\sharp & 0 & \cdots \\ e_0^\flat & e_1^\natural & e_1^\sharp & \\ 0 & e_1^\flat & e_2^\natural & \ddots \\ \vdots & & \ddots & \ddots \end{bmatrix}, \text{ where}$$

$$\begin{aligned} d_n^\natural := d_n^\natural(a,b,c,d) &= \frac{q^{n-1}}{(1-q^{2n-2}abcd)(1-q^{2n}abcd)} \\ &\quad\times[bd(a+c)+(b+d)q-abcd(b+d)q^{n-1}-\{bd(a+c)+abcd(b+d)\}q^n \\ &\quad -bd(a+c)q^{n+1}+ab^2cd^2(a+c)q^{2n-1}+abcd(b+d)q^{2n}], \\ e_n^\natural := e_n^\natural(a,b,c,d) &= \frac{q^{n-1}}{(1-q^{2n-2}abcd)(1-q^{2n}abcd)} \\ &\quad\times[ac(b+d)+(a+c)q-abcd(a+c)q^{n-1}-\{ac(b+d)+abcd(a+c)\}q^n \\ &\quad -ac(b+d)q^{n+1}+a^2bc^2d(b+d)q^{2n-1}+abcd(a+c)q^{2n}], \end{aligned}$$

$$\begin{aligned} d_n^\sharp := d_n^\sharp(a,b,c,d) &= \frac{1}{1-q^nac}\mathcal{A}_n, & e_n^\sharp := e_n^\sharp(a,b,c,d) &= -\frac{q^nac}{1-q^nac}\mathcal{A}_n, \\ d_n^\flat := d_n^\flat(a,b,c,d) &= -\frac{q^nbd}{1-q^nbd}\mathcal{A}_n, & e_n^\flat := e_n^\flat(a,b,c,d) &= \frac{1}{1-q^nbd}\mathcal{A}_n, \text{ and} \end{aligned}$$

$$\begin{aligned} \mathcal{A}_n :=& \mathcal{A}_n(a,b,c,d) \\ =& \left[\frac{(1-q^{n-1}abcd)(1-q^{n+1})(1-q^nab)(1-q^nac)(1-q^nad)(1-q^nbc)(1-q^nbd)(1-q^ncd)}{(1-q^{2n-1}abcd)(1-q^{2n}abcd)^2(1-q^{2n+1}abcd)}\right]^{1/2}. \end{aligned}$$

Remark 2.4. *These matrices have the property that the coefficients in the nth row of $\mathsf{d}+\mathsf{e}$ are the coefficients in the three-term recurrence for the Askey-Wilson polynomials (2.5). That three-term recurrence is given by*

$$A_nP_{n+1}(x)+B_nP_n(x)+C_nP_{n-1}(x)=2xP_n(x), \tag{2.5}$$

with $P_0(x) = 1$ and $P_{-1}(x) = 0$, where

$$A_n = \frac{1 - q^{n-1}abcd}{(1 - q^{2n-1}abcd)(1 - q^{2n}abcd)},$$

$$B_n = \frac{q^{n-1}}{(1 - q^{2n-2}abcd)(1 - q^{2n}abcd)}[(1 + q^{2n-1}abcd)(qs + abcds') - q^{n-1}(1 + q)abcd(s + qs')],$$

$$C_n = \frac{(1 - q^n)(1 - q^{n-1}ab)(1 - q^{n-1}ac)(1 - q^{n-1}ad)(1 - q^{n-1}bc)(1 - q^{n-1}bd)(1 - q^{n-1}cd)}{(1 - q^{2n-2}abcd)(1 - q^{2n-1}abcd)},$$

$$s = a + b + c + d, \qquad s' = a^{-1} + b^{-1} + c^{-1} + d^{-1}.$$

It's now a direct consequence of Theorem 2.2 and Remark 2.3 that the nth moment of the Askey-Wilson polynomials is given by $\langle \widetilde{W}|(\mathsf{d} + \mathsf{e})^n|\widetilde{V}\rangle$.

Also define

$$D_n^\natural = d_n^\natural\left(\frac{a}{\sqrt{y}}, b\sqrt{y}, \frac{c}{\sqrt{y}}, d\sqrt{y}\right), \qquad E_n^\natural = e_n^\natural\left(\frac{a}{\sqrt{y}}, b\sqrt{y}, \frac{c}{\sqrt{y}}, d\sqrt{y}\right),$$
$$D_n^\sharp = d_n^\sharp\left(\frac{a}{\sqrt{y}}, b\sqrt{y}, \frac{c}{\sqrt{y}}, d\sqrt{y}\right), \qquad E_n^\sharp = e_n^\sharp\left(\frac{a}{\sqrt{y}}, b\sqrt{y}, \frac{c}{\sqrt{y}}, d\sqrt{y}\right),$$
$$D_n^\flat = d_n^\flat\left(\frac{a}{\sqrt{y}}, b\sqrt{y}, \frac{c}{\sqrt{y}}, d\sqrt{y}\right), \qquad E_n^\flat = e_n^\flat\left(\frac{a}{\sqrt{y}}, b\sqrt{y}, \frac{c}{\sqrt{y}}, d\sqrt{y}\right).$$

Lemma 2.5. $yd_n^\natural + e_n^\natural = \sqrt{y}(D_n^\natural + E_n^\natural)$.

Proof. This follows from the fact that $D_n^\natural = \sqrt{y}d_n^\natural$ and $E_n^\natural = \frac{1}{\sqrt{y}}e_n^\natural$. □

Lemma 2.6. $(yd_n^\sharp + e_n^\sharp)(yd_n^\flat + e_n^\flat) = y(D_n^\sharp + E_n^\sharp)(D_n^\flat + E_n^\flat)$.

Proof. First observe that

$$y(D_n^\sharp + E_n^\sharp) = \frac{y - q^n ac}{1 - q^n acy^{-1}}\mathcal{A}_n\left(\frac{a}{\sqrt{y}}, b\sqrt{y}, \frac{c}{\sqrt{y}}, d\sqrt{y}\right),$$
$$D_n^\flat + E_n^\flat = \frac{1 - q^n bdy}{1 - q^n bdy}\mathcal{A}_n\left(\frac{a}{\sqrt{y}}, b\sqrt{y}, \frac{c}{\sqrt{y}}, d\sqrt{y}\right), \text{ and}$$
$$\mathcal{A}_n\left(\frac{a}{\sqrt{y}}, b\sqrt{y}, \frac{c}{\sqrt{y}}, d\sqrt{y}\right) = \frac{(1 - q^n acy^{-1})(1 - q^n bdy)}{(1 - q^n ac)(1 - q^n bd)}^{1/2}\mathcal{A}_n(a, b, c, d).$$

Multiplying the first two equations gives

$$y(D_n^\sharp + E_n^\sharp)(D_n^\flat + E_n^\flat) = \frac{(y - q^n ac)(1 - q^n bdy)}{(1 - q^n acy^{-1})(1 - q^n bdy)}\left(\mathcal{A}_n\left(\frac{a}{\sqrt{y}}, b\sqrt{y}, \frac{c}{\sqrt{y}}, d\sqrt{y}\right)\right)^2.$$

Then using the third equation we have

$$y(D_n^\sharp + E_n^\sharp)(D_n^\flat + E_n^\flat) = \frac{(y - q^n ac)(1 - q^n bdy)}{(1 - q^n ac)(1 - q^n bd)}\mathcal{A}_n^2.$$

It remains to see that

$$(yd_n^\sharp + e_n^\sharp)(yd_n^\flat + e_n^\flat) = \frac{(y - q^n ac)(1 - q^n bdy)}{(1 - q^n ac)(1 - q^n bd)}\mathcal{A}_n^2,$$

but this follows from the definition of $d_n^\sharp, e_n^\sharp, d_n^\flat, e_n^\flat$.

□

We now define matrices $\widetilde{D}$ and $\widetilde{E}$ by

$$\widetilde{D} = \frac{1}{1-q}(1+\mathsf{d}), \qquad \widetilde{E} = \frac{1}{1-q}(1+\mathsf{e}). \tag{2.6}$$

Then $y\widetilde{D} + \widetilde{E} = \frac{1}{1-q}(1 + y1 + y\mathsf{d} + \mathsf{e})$.

Proposition 2.7. *The nth moment of the specialization of the Askey-Wilson polynomials $P_m(x + \frac{\frac{1}{\sqrt{y}}+\sqrt{y}}{2}; \frac{a}{\sqrt{y}}, b\sqrt{y}, \frac{c}{\sqrt{y}}, d\sqrt{y}|q)$ is equal to*

$$\frac{\langle \widetilde{W}|(y\widetilde{D} + \widetilde{E})^n|\widetilde{V}\rangle(1-q)^n}{2^n\sqrt{y}^n}.$$

Proof. Note that

$$\langle \widetilde{W}|(y\widetilde{D} + \widetilde{E})^n|\widetilde{V}\rangle = \frac{1}{(1-q)^n}\langle \widetilde{W}|(1 + y1 + y\mathsf{d} + \mathsf{e})^n|\widetilde{V}\rangle,$$

and so

$$\langle \widetilde{W}|(y\widetilde{D} + \widetilde{E})^n|\widetilde{V}\rangle(1-q)^n\sqrt{y}^{-n} = \langle \widetilde{W}|(\frac{1}{\sqrt{y}}1 + \sqrt{y}1 + \sqrt{y}\mathsf{d} + \frac{1}{\sqrt{y}}\mathsf{e})^n|\widetilde{V}\rangle. \tag{2.7}$$

It's easy to see that the right-hand-side of equation (2.7) is the nth moment for the monic polynomials $q_m(x + \frac{1}{\sqrt{y}} + \sqrt{y})$, where the q_m's are defined by the three-term recurrence $xq_n(x) = q_{n+1}(x) + B'_n q_n(x) + A'_{n-1}C'_n q_{n-1}(x)$, and where A'_n, B'_n, C'_n are given by the nth row of the tridiagonal matrix $\sqrt{y}\mathsf{d} + \frac{1}{\sqrt{y}}\mathsf{e}$.

Alternatively, letting $Q_n(x) = q_n(2x)$, we can interpret the right-hand-side of (2.7) as 2^n times the nth moment for the non-monic polynomials $Q_m(x + \frac{\frac{1}{\sqrt{y}}+\sqrt{y}}{2})$, which are defined by the recurrence

$$2xQ_n(x) = Q_{n+1}(x) + B'_n Q_n(x) + A'_{n-1}C'_n Q_{n-1}(x).$$

By Lemmas 2.5 and 2.6,

$$B'_n = \frac{yd^\natural_n + e^\natural_n}{\sqrt{y}} = D^\natural_n + E^\natural_n, \text{ and}$$

$$A'_{n-1}C'_n = \frac{yd^\sharp_n + e^\sharp_n}{\sqrt{y}}\frac{yd^\flat_n + e^\flat_n}{\sqrt{y}} = (D^\sharp_n + E^\sharp_n)(D^\flat_n + E^\flat_n).$$

Note also that by Remark 2.3, the polynomials defined by $2xQ_n(x) = Q_{n+1}(x) + B'_n Q_n(x) + A'_{n-1}C'_n Q_{n-1}(x)$ and the polynomials defined by $2xQ'_n(x) = A'_n Q'_{n+1}(x) + B'_n Q'_n(x) + C'_n Q'_{n-1}(x)$ have the same moments.

Therefore by Remark 2.4, the nth moment of the polynomials $Q_m(x)$ is the nth moment of the Askey-Wilson polynomials $P_m(x)$, with the specialization $a \to \frac{a}{\sqrt{y}}$, $b \to b\sqrt{y}$, $c \to \frac{c}{\sqrt{y}}$, and $d \to d\sqrt{y}$. The proposition follows. □

Now we would like to relate this to the matrices $D, E, |V\rangle, \langle W|$ given in [12, Definition 6.1], which have a combinatorial interpretation in terms of staircase tableaux. We do not need their definitions here, but only the following property.

Theorem 2.8. [11, 12, Corteel, Williams] *We have that*

$$Z_n(y;\alpha,\beta,\gamma,\delta;q) = \langle W|(yD+E)^n|V\rangle.$$

Additionally, the coefficient of y^i above is proportional to the probability that in the asymmetric exclusion process on n sites, exactly i sites are occupied by a particle.

The following result gives an explicit relation between the moments of the Askey-Wilson polynomials and the fugacity partition function of the ASEP.

Corollary 2.9. *The nth moment of the specialization of the Askey-Wilson polynomials $P_m(2\sqrt{y}x+1+y; \frac{a}{\sqrt{y}}, b\sqrt{y}, \frac{c}{\sqrt{y}}, d\sqrt{y}|q)$ is equal to*

$$\frac{(1-q)^n}{\prod_{j=0}^{n-1}(\alpha\beta-\gamma\delta q^j)}\langle W|(yD+E)^n|V\rangle = \frac{(1-q)^n}{\prod_{j=0}^{n-1}(\alpha\beta-\gamma\delta q^j)} Z_n(y;\alpha,\beta,\gamma,\delta;q),$$

where $\alpha,\beta,\gamma,\delta$ are given by (1.2).

Proof. The matrices $\widetilde{D},\widetilde{E},\widetilde{V},\widetilde{W}$ satisfy the Matrix Ansatz of Derrida-Evans-Hakim-Pasquier, see [12, Theorem 5.1]. On the other hand, the matrices D,E,V,W satisfy the modified Matrix Ansatz of [12, Theorem 5.2]. By [12, Lemma 7.1] and the proof of [12, Theorem 4.1], these can be related via

$$\langle\widetilde{W}|(y\widetilde{D}+\widetilde{E})^n|\widetilde{V}\rangle = \frac{\langle W|(yD+E)^n|V\rangle}{\prod_{j=0}^{n-1}(\alpha\beta-\gamma\delta q^j)}.$$

Now the proof follows from Proposition 2.7,[1] together with the observation that $2^n\sqrt{y}^n$ times the nth moment of the polynomials $P_m(x+\frac{\frac{1}{\sqrt{y}}+\sqrt{y}}{2}; \frac{a}{\sqrt{y}}, b\sqrt{y}, \frac{c}{\sqrt{y}}, d\sqrt{y}|q)$ is equal to the nth moment of the polynomials $P_m(2\sqrt{y}x+1+y; \frac{a}{\sqrt{y}}, b\sqrt{y}, \frac{c}{\sqrt{y}}, d\sqrt{y}|q)$. ☐

Corollary 2.9 is equivalent to the following one:

Corollary 2.10. *The fugacity partition function $Z_n(y;\alpha,\beta,\gamma,\delta;q)$ is equal to*

$$(abcd)_n\sqrt{y}^n(\alpha\beta)^n \times \mu_n$$

where μ_n are the moments of the orthogonal polynomials defined by

$$(x-b_n)G_n(x) = G_{n+1}(x) + \lambda_n G_{n-1}(x)$$

where

$$b_n = \frac{1/\sqrt{y}+\sqrt{y}+B_n}{1-q} \quad \text{and} \quad \lambda_n = \frac{A_{n-1}C_n}{(1-q)^2}$$

and A_n, B_n, C_n are the coefficients of the 3-term recurrence of the Askey Wilson given in (2.5) *with $a\to a/\sqrt{y}$, $b\to b\sqrt{y}$, $c\to c/\sqrt{y}$, $d\to d\sqrt{y}$ and a,b,c,d are given by* (1.4).

If we set $y=-1$ in Corollary 2.9 and perform a simple change of variables taking $-ai\to a, bi\to b, -ci\to c, di\to d$, we give a formula for the Askey-Wilson moments "on the nose."

[1]Note that Uchiyama-Sasamoto-Wadati [29] defined vectors $\widetilde{W}$ and $\widetilde{V}$ to be $h_0^{1/2}(1,0,0,\cdots)$ and $h_0^{1/2}(1,0,0,\cdots)^T$, not $(1,0,0,\cdots)$ and $(1,0,0,\cdots)^T$ as we have done here. However, their weight function w did not have the factor of h_0 as ours has, and these two discrepancies "cancel each other out."

Corollary 2.11. *The nth moment of the Askey-Wilson polynomials $P_m(x;a,b,c,d|q)$ is equal to*

$$\mu_n(a,b,c,d|q) = \frac{(1-q)^n}{2^n i^n \prod_{j=0}^{n-1}(\alpha\beta - \gamma\delta q^j)} \langle W|(-D+E)^n|V\rangle,$$

where $\alpha,\beta,\gamma,\delta$ are given by (1.3).

This finishes the proof of Theorem 1.11, because Theorem 1.11 is equivalent to Corollary 2.11 by setting $y=-1$ in Theorem 2.8.

3. Explicit formulae for Askey-Wilson moments and staircase tableaux

In this section we will give some explicit formulas for the moments of the Askey-Wilson polynomials. We first prove a more general statement. Recall Remark 1.6.

Proposition 3.1. *Let $p(x)$ be a degree n polynomial in x. Then*

$$\oint_C \frac{p(x)w(x,a,b,c,d|q)dz}{4\pi i z} = \sum_{k=0}^{n} \frac{(ab,ac,ad;q)_k}{(abcd;q)_k} q^k \sum_{j=0}^{k} \frac{q^{-j^2}a^{-2j}p(\frac{q^j a + q^{-j}/a}{2})}{(q,q^{-2j+1}/a^2;q)_j (q,a^2q^{1+2j};q)_{k-j}}.$$

Recall the Askey-Wilson weight function w defined in Section 1. Let $\phi_n(x;a) = (ae^{i\theta}, ae^{-i\theta};q)_n$; this is a polynomial in x of degree n. Note that

$$w(x,a,b,c,d|q)\phi_n(x;a) = w(x,aq^n,b,c,d|q)\frac{h_0(aq^n,b,c,d,q)}{h_0(a,b,c,d,q)}.$$

Therefore

Lemma 3.2.

$$\oint_C \frac{\phi_n(x;a)w(x,a,b,c,d|q)dz}{4\pi i z} = \frac{h_0(aq^n,b,c,d,q)}{h_0(a,b,c,d,q)}.$$

Our strategy for proving Proposition 3.1 will be to expand $f(x)$ in the basis $\phi_n(x;a)$ by using a result of Ismail and the third author [19], and then to apply Lemma 3.2.

Theorem 3.3. [19, Theorem 1.1] *If we write the degree n polynomial $p(x)$ as $\sum_{k=0}^{n} p_k\phi_k(x;a)$ then*

$$p_k = \frac{(q-1)^k}{(2a)^k(q;q)_k} q^{-\frac{k(k-1)}{4}} (D_q^k p)(x_k),$$

where

$$(D_q^k p)(x) = \frac{2^k q^{\frac{k(1-k)}{4}}}{(q^{1/2}-q^{-1/2})^k} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{q^{j(k-j)} z^{2j-k} \breve{p}(q^{(k-2j)/2}z)}{(q^{1+k-2j}z^2;q)_j (q^{1-k+2j}z^{-2};q)_{k-j}},$$

$x_k = (aq^{k/2} + a^{-1}q^{-k/2})/2$, $x=\cos\theta$, $z=e^{i\theta}$, $\begin{bmatrix} k \\ j \end{bmatrix}_q = \frac{(q;q)_k}{(q;q)_j(q;q)_{k-j}}$, *and* $\breve{p}(x) = f(\frac{x+x^{-1}}{2})$.

We can now prove Proposition 3.1.

Proof. Write $p(x)=\sum_{k=0}^n p_k\phi_k(x;a)$. Then when $x_k=(aq^{k/2}+a^{-1}q^{-k/2})/2$, we get $z_k=aq^{k/2}$ and

$$\begin{aligned}
p_k &= \frac{(q-1)^k}{(2a)^k(q;q)_k}q^{-\frac{k(k-1)}{4}}\frac{2^kq^{\frac{k(1-k)}{4}}}{(q^{1/2}-q^{-1/2})^k}\sum_{j=0}^{k}\begin{bmatrix} k\\ j\end{bmatrix}_q\frac{q^{j(k-j)}(aq^{k/2})^{2j-k}\check{p}(aq^{k-j})}{(q^{1+2k-2j}a^2;q)_j(q^{-1-2k+2j}a^{-2};q)_{k-j}}\\
&= \frac{q^kq^{k(1-k)/2}}{a^k}\sum_{j=0}^{k}\frac{1}{(q;q)_j(q;q)_{k-j}}\frac{q^{j(k-j)}(aq^{k/2})^{2j-k}\check{p}(aq^{k-j})}{(q^{1+2k-2j}a^2;q)_j(q^{-1-2k+2j}a^{-2};q)_{k-j}}\\
&= q^k\sum_{j=0}^{k}\frac{q^{-(k-j)^2}a^{2j-2k}\check{p}(aq^{k-j})}{(q,q^{1+2k-2j}a^2;q)_j(q,q^{-1-2k+2j}a^{-2};q)_{k-j}}.
\end{aligned}$$

Note that

$$\frac{h_0(aq^k,b,c,d,q)}{h_0(a,b,c,d,q)}=\frac{(ab,ac,ad;q)_k}{(abcd;q)_k}.$$

Therefore

$$\begin{aligned}
\oint_C\frac{p(x)w(x,a,b,c,d|q)dz}{4\pi iz} &= \sum_{k=0}^{n}p_k\frac{h_0(aq^k,b,c,d,q)}{h_0(a,b,c,d,q)}\\
&= \sum_{k=0}^{n}\frac{(ab,ac,ad;q)_k}{(abcd;q)_k}q^k\sum_{j=0}^{k}\frac{q^{-(k-j)^2}a^{2j-2k}p((aq^{k-j}+a^{-1}q^{j-k})/2)}{(q,q^{1+2k-2j}a^2;q)_j(q,q^{1-2k+2j}a^{-2};q)_{k-j}}\\
&= \sum_{k=0}^{n}\frac{(ab,ac,ad;q)_k}{(abcd;q)_k}q^k\sum_{j=0}^{k}\frac{q^{-j^2}a^{-j}p((aq^j+a^{-1}q^{-j})/2)}{(q,q^{1+2j}a^2;q)_{k-j}(q,q^{1-2j}a^{-2};q)_j}.
\end{aligned}$$

□

We can now prove Theorem 1.12.

Proof. Setting $p(x)=x^n$ in Proposition 3.1, we obtain

$$\mu_n(a,b,c,d|q)=\frac{1}{2^n}\sum_{k=0}^{n}\frac{(ab,ac,ad;q)_k}{(abcd;q)_k}q^k\times\sum_{j=0}^{k}q^{-j^2}a^{-2j}\frac{(q^ja+q^{-j}/a)^n}{(q,q^{-2j+1}/a^2;q)_j(q,a^2q^{1+2j};q)_{k-j}}.$$

□

Example 3.4.

$$
\begin{aligned}
\mu_1(a,b,c,d) =&(-a-b-c-d+abc+abd+acd+bcd)/(2(-1+abcd))\\
\mu_2(a,b,c,d) =&(1+a^2+ab+b^2+ac+bc-a^2bc-ab^2c+c^2-abc^2+ad\\
&+bd-a^2bd-ab^2d+cd-a^2cd-4abcd-b^2cd+a^2b^2cd\\
&-ac^2d-bc^2d+a^2bc^2d+ab^2c^2d+d^2-abd^2-acd^2-bcd^2\\
&+a^2bcd^2+ab^2cd^2+abc^2d^2-a^2b^2c^2d^2-q+abq+acq+bcq\\
&-a^2bcq-ab^2cq-abc^2q+a^2b^2c^2q+adq+bdq-a^2bdq\\
&-ab^2dq+cdq-a^2cdq-4abcdq-b^2cdq+a^2b^2cdq-ac^2dq\\
&-bc^2dq+a^2bc^2dq+ab^2c^2dq-abd^2q+a^2b^2d^2q\\
&-acd^2q-bcd^2q+a^2bcd^2q+ab^2cd^2q+a^2c^2d^2q+abc^2d^2q\\
&+b^2c^2d^2q+a^2b^2c^2d^2q)/(4(-1+abcd)(-1+abcdq)).
\end{aligned}
$$

We also use Proposition 3.1 to prove Theorem 1.13.

Proof. Now we use the result of Corollary 2.9. To get the fugacity partition function of the ASEP or equivalently the generating polynomial of staircase tableaux, we have to take $p(x) = (1+y+2\sqrt{y}x)^n$ and substitute

$$a \to a/\sqrt{y}, \qquad b \to b\sqrt{y}, \qquad c \to c/\sqrt{y}, \qquad d \to d\sqrt{y}$$

in Proposition 3.1. □

Example 3.5.

$$
\begin{aligned}
Z_1 =&\alpha y+\delta y+\beta+\gamma\\
Z_2 =&\alpha^2y^2+\alpha\delta y^2+\alpha^2\delta y^2+\alpha\beta\delta y^2+\alpha\delta^2y^2+\alpha\delta\gamma y^2+\alpha\delta qy^2+\delta^2qy^2+\alpha\beta y+\alpha^2\beta y+\alpha\beta^2y+\\
&\beta\delta y+\alpha\beta\delta y+\alpha\gamma y+\alpha\beta\gamma y+\delta\gamma y+\alpha\delta\gamma y+\beta\delta\gamma y+\delta^2\gamma y+\delta\gamma^2y+\alpha\beta qy+\beta\delta qy+\\
&\alpha\gamma qy+\delta\gamma qy+\beta^2+\beta\gamma+\alpha\beta\gamma+\beta^2\gamma+\beta\delta\gamma+\beta\gamma^2+\beta\gamma q+\gamma^2q.
\end{aligned}
$$

3.1. **Askey Wilson moments and the partition function when $q = 0$.** If $q = 0$ the moments may be computed in another way, using a contour integral and the residue calculus. Recall the substitutions from Remark 1.6.

Proposition 3.6. *Let $p(x)$ be any polynomial in x, and let $f(z,a,b,c,d) = (1-az)(1-a/z)(1-bz)(1-b/z)(1-cz)(1-c/z)(1-dz)(1-d/z)$. Then*

$$\begin{aligned}
\oint_C \frac{p(x)w(x,a,b,c,d|0)dz}{4\pi iz} &= \frac{-1}{2}\frac{(1-ab)(1-ac)(1-ad)(1-bc)(1-bd)(1-cd)}{1-abcd} \\
&\times\Bigg(\frac{p(\frac{a+1/a}{2})(a-1/a)^2}{(1-a^2)(1-ab)(1-b/a)(1-ca)(1-c/a)(1-da)(1-d/a)} \\
&+\frac{p(\frac{b+1/b}{2})(b-1/b)^2}{(1-b^2)(1-ab)(1-a/b)(1-cb)(1-c/b)(1-db)(1-d/b)} \\
&+\frac{p(\frac{c+1/c}{2})(c-1/c)^2}{(1-c^2)(1-ac)(1-a/c)(1-cb)(1-b/c)(1-dc)(1-d/c)} \\
&+\frac{p(\frac{d+1/d}{2})(d-1/d)^2}{(1-d^2)(1-ad)(1-a/d)(1-db)(1-b/d)(1-dc)(1-c/d)} \\
&+Res\Bigg(\frac{p(\frac{z+1/z}{2})(z-1/z)^2}{zf(z,a,b,c,d)}, z=0\Bigg)\Bigg)
\end{aligned}$$

Proof. Assume that $|a|,|b|,|c|,|d|<1$; these conditions are not necessary later. Using the Cauchy Residue Theorem, we get

$$\oint_C \frac{p(x)w(x,a,b,c,d|0)dz}{4\pi iz} = \frac{1}{2}\sum_k Res\left(\frac{p(x)w(x,a,b,c,d|0)}{z}, z=a_k\right), \tag{3.8}$$

where the a_k are the poles inside C.

Note that at $q=0$, we have

$$\begin{aligned}
h_0(a,b,c,d,0) &= \frac{1-abcd}{(1-ab)(1-ac)(1-ad)(1-bc)(1-bd)(1-cd)}, \text{ and} \\
w(\cos\theta,a,b,c,d|0) &= \frac{-(z-1/z)^2}{h_0(a,b,c,d,0)f(z,a,b,c,d)}.
\end{aligned}$$

There are five poles inside C: $z=a,b,c,d$ and 0. Substituting into (3.8) gives the result. $\square$

Let $H_n(a,b,c,d)$ be the homogeneous symmetric function of degree n in the 8 variables a, b, c, d, $1/a$, $1/b$, $1/c$, $1/d$.

Theorem 3.7. *The partition function $Z_n(y;\alpha,\beta,\gamma,\delta;0)$ is*

$$\begin{aligned}
&\frac{-(\alpha\beta)^n}{2}(1-AB)(1-AC)(1-AD)(1-BC)(1-BD)(1-CD)\\
&\times\Bigg(\frac{(1+(1/A+A)\sqrt{y}+y)^n(A-1/A)^2}{(1-A^2)(1-AB)(1-B/A)(1-CA)(1-C/A)(1-DA)(1-D/A)}\\
&+\frac{(1+(1/B+B)\sqrt{y}+y)^n(B-1/B)^2}{(1-B^2)(1-AB)(1-A/B)(1-CB)(1-C/B)(1-DB)(1-D/B)}\\
&+\frac{(1+(1/C+C)\sqrt{y}+y)^n(C-1/C)^2}{(1-C^2)(1-AC)(1-A/C)(1-CB)(1-B/C)(1-DC)(1-D/C)}\\
&+\frac{(1+(1/D+D)\sqrt{y}+y)^n(D-1/D)^2}{(1-D^2)(1-AD)(1-A/D)(1-DB)(1-B/D)(1-DC)(1-C/D)}\\
&+\frac{1}{ABCD}\sum_{j=0}^{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n}{j}\sqrt{y}^{n+k-j}\\
&\times(H_{n-2-k-j}(A,B,C,D)-2H_{n-4-k-j}(A,B,C,D)+H_{n-6-k-j}(A,B,C,D)\Bigg)
\end{aligned}$$

where $A=a/\sqrt{y}$, $B=b\sqrt{y}$, $C=c/\sqrt{y}$, $D=d\sqrt{y}$ and a,b,c,d as in Proposition 1.13.

Proof. Use Proposition 3.6 with $p(x)=(1+y+2\sqrt{y}x)^n$. We need to compute the residue of

$$\frac{p(\frac{z+1/z}{2})(z-1/z)^2}{zf(z,a,b,c,d)}$$

at $z=0$ with $f(z,a,b,c,d)=(1-az)(1-a/z)(1-bz)(1-b/z)(1-cz)(1-c/z)(1-dz)(1-d/z)$. Now we substitute $a\to a/\sqrt{y}$, $b\to b\sqrt{y}$, $c\to c/\sqrt{y}$, $d\to d\sqrt{y}$. Since

$$\frac{1}{f(z,A,B,C,D)}=z^4/ABCD\sum_{s=0}^{\infty}H_s(A,B,C,D)z^s,$$

we need the residue of

$$\frac{(\sqrt{y}(z+1/z)+y+1)^n(z-1/z)^2z^3}{ABCD}\sum_{s=0}^{\infty}H_s(A,B,C,D)z^s$$

at $z=0$ or equivalently the coefficient of z^n in

$$\frac{(\sqrt{y}+z)^n(1+\sqrt{y}z)^n(z-1/z)^2z^4}{ABCD}\sum_{s=0}^{\infty}H_s(A,B,C,D)z^s,$$

which is

$$\sum_{k=0}^{n}\binom{n}{k}\sqrt{y}^{n-k}\sum_{j=0}^{n}\binom{n}{j}\sqrt{y}^{j}(H_{n-2-k-j}-2H_{n-4-k-j}+H_{n-6-k-j}).$$

□

Example 3.8.

$$\begin{aligned}Z_2(y;\alpha,\beta,\gamma,\delta;0) = {} & y^2\alpha^2 + y^2\alpha\delta + y^2\alpha^2\delta + y^2\alpha\beta\delta + y^2\alpha\delta^2 + y^2\alpha\delta\gamma + y\alpha\beta + y\alpha^2\beta \\ & + y\alpha\beta^2 + y\beta\delta + y\alpha\beta\delta + y\alpha\gamma + y\alpha\beta\gamma + y\delta\gamma + y\alpha\delta\gamma + y\beta\delta\gamma \\ & + y\delta^2\gamma + y\delta\gamma^2 + \beta^2 + \beta\gamma + \alpha\beta\gamma + \beta^2\gamma + \beta\delta\gamma + \beta\gamma^2.\end{aligned}$$

The following corollaries below follows directly from Theorem 3.7.

Corollary 3.9. *When $\alpha = \beta = 1$ and $\gamma = \delta = q = 0$, the generating function for the numbers $Z_n(y) = Z_n(y;\alpha,\beta,\gamma,\delta;q)$ is*

$$\sum_{n=0}^{\infty} Z_n(y)w^n = (1+t)(1+yt)$$

where $w = \frac{t}{(1+t)(1+yt)}$. Note that this is the generating function of the Narayana numbers [32].

Corollary 3.10. *When $\alpha = \beta = \gamma = 1$ and $\delta = q = 0$, the generating function for the numbers $Z_n(y) = Z_n(y;\alpha,\beta,\gamma,\delta;q)$ is*

$$\sum_{n=0}^{\infty} Z_n(y)w^n = \frac{(1+t)(1+yt)}{(1-t-t^2)},$$

where $w = \frac{t}{(1+t)(1+yt)}$. Note that when $y = 1$, $Z_n(y)$ is the sequence Sloane A026671, and when $y = 0$ $Z_n(y)$ is the sequence odd Fibonacci numbers.

Example 3.11. *Set $\alpha = \beta = \gamma = 1$ and $\delta = q = 0$. The polynomials $Z_n(y)$ are*

$$\begin{aligned}Z_0 &= 1 \\ Z_1 &= 2 + y \\ Z_2 &= 5 + 5y + y^2 \\ Z_3 &= 13 + 20y + 9y^2 + y^3 \\ Z_4 &= 34 + 72y + 52y^2 + 14y^3 + y^4 \\ Z_5 &= 89 + 242y + 245y^2 + 110y^3 + 20y^4 + y^5\end{aligned}$$

Corollary 3.12. *When $\alpha = \beta = \gamma = \delta = 1$ and $q = 0$, the generating function for the numbers $Z_n(y) = Z_n(y;\alpha,\beta,\gamma,\delta;q)$ is*

$$\sum_{n=1}^{\infty} Z_n(y)w^n = \frac{2(1+y)t(1+t)(1+yt)}{(1-t-t^2)(1-yt-y^2t^2)},$$

where $w = \frac{t}{(1+t)(1+yt)}$.

Example 3.13. *Set $\alpha = \beta = \gamma = \delta = 1$ and $q = 0$. The polynomials $Z_n(y)$ are*

$$\begin{aligned}Z_1 &= 2(1+y) \\ Z_2 &= 6(1+y)^2 \\ Z_3 &= 2(1+y)(8 + 15y + 8y^2) \\ Z_4 &= 2(1+y)^2(21 + 34y + 21y^2) \\ Z_5 &= 2(1+y)(55 + 181y + 253y^2 + 181y^3 + 55y^4)\end{aligned}$$

Remark 3.14. *Corollaries 3.9, 3.10 and 3.12 can also be proved by induction. We can write recurrences for $Z_{n,k,j}$ the number of tableaux of size n with k rows indexed by α or γ, and j entries equal to α or δ on the diagonal. From this we write a functional equation for $Z(w,t,y) = \sum_{n,k,j} Z_{n,k,j} w^n t^k y^j$. Finally t is used as a catalytic variable and we extract $Z(w,1,y)$.*

4. Combinatorics of staircase tableaux

The motivation for defining staircase tableaux in [11, 12] was to give a combinatorial formula for the stationary distribution of the ASEP with all parameters $\alpha, \beta, \gamma, \delta, q$ general. Such a formula had already been given in [9] using permutation tableaux, when $\gamma = \delta = 0$. Therefore it follows that the set of staircase tableaux containing only α's or β's are in bijection with both the permutation tableaux coming from Postnikov's work [26, 28], and the alternative tableaux introduced by Viennot [31]. These bijections are explained in [12, Section 9]. As a consequence, the staircase tableaux of size n with only α's and β's are in bijection with permutations on $n+1$ letters [28, 8, 4]. Moreover, one can interpret the parameter q as counting the number of *crossings* or the number of patterns $31-2$ in the permutation [6, 28].

In this section we explore more of the combinatorial properties of staircase tableaux, and in particular, explain the formulas in Table 1.

4.1. Enumeration of staircase tableaux when $q = 1$.

As before, we set $Z_n = \sum_{\mathcal{T}} \mathrm{wt}(\mathcal{T})$, where the sum is over all staircase tableaux of size n. When $q = y = 1$, the weighted sum of staircase tableaux of size n factors as a product of n terms.

Theorem 4.1. *When $q = y = 1$,*

$$Z_n(1;\alpha,\beta,\gamma,\delta;1) = \prod_{j=0}^{n-1} (\alpha+\beta+\gamma+\delta+j(\alpha+\gamma)(\beta+\delta)).$$

Proof. When $q = y = 1$, it's clear from the definition of staircase tableaux that $Z_n(1;\alpha,\beta,\gamma,\delta;1) = Z_n(1;\alpha+\gamma,\beta+\delta,0,0;1)$. The result then follows from the fact that

$$Z_n(1;\alpha,\beta,0,0;1) = \prod_{j=0}^{n-1} (\alpha+\beta+j\alpha\beta), \tag{4.9}$$

which was proved combinatorially (using the language of permutation tableaux) in [8]. We will give another proof of Equation 4.9 in Section 4.2. □

Remark 4.2. *Note that Theorem 4.1 and Theorem 1.5 immediately imply a result of Uchiyama, Sasamoto, and Wadati [29], which is that the partition function of the ASEP with $\alpha, \beta, \gamma, \delta$ general and $q = 1$ is given by*

$$\prod_{j=0}^{n-1} (\alpha+\beta+\gamma+\delta+j(\alpha+\gamma)(\beta+\delta)).$$

They prove this result by noting that when $q = 1$, the partition function Z_n of the ASEP is equal to the nth moment of the Laguerre polynomials $L_n^{(\lambda)}(x)$ with

$$\lambda = \frac{\alpha+\beta+\gamma+\delta}{(\alpha+\gamma)(\beta+\delta)} - 1,$$

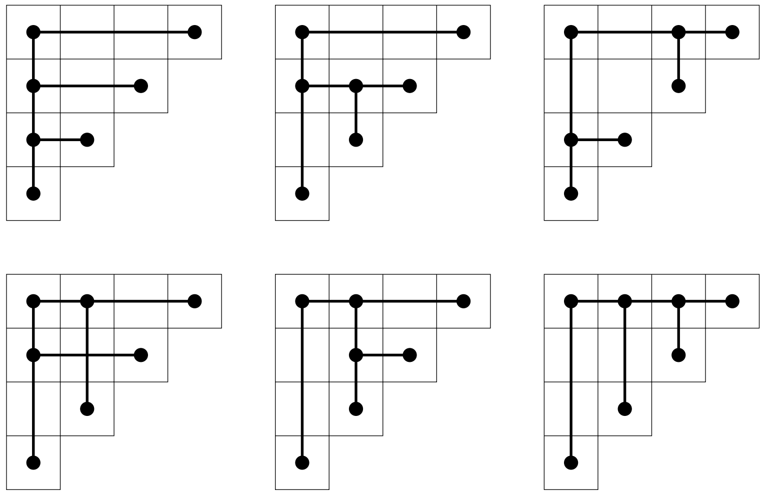

FIGURE 2. The staircase trees of size 4

defined by $L_0^{(\lambda)}(x) = 1$, $L_{-1}^{(\lambda)}(x) = 0$, *and*

$$(n+1)L_{n+1}^{(\lambda)}(x) - (2n+\lambda+1-x)L_n^{(\lambda)}(x) + (n+\lambda)L_{n-1}^{(\lambda)}(x) = 0.$$

4.2. **Staircase tableaux and trees.** We begin by describing a bijective approach to understanding staircase tableaux that uses an underlying forest structure of the tableaux.

Let $D(\mathcal{T})$ be the diagram of a staircase tableau of α's and β's. Regard the entries α and β as vertices of a graph. For each nondiagonal vertex v, regard the two nearest vertices directly to the right and directly below v as the children of v, called the *row child* and *column child*, respectively. In this way $D(\mathcal{T})$ becomes a *complete rooted binary forest*, i.e., a forest for which every component is a rooted tree, and every non-endpoint vertex has exactly two children. The endpoints are just the diagonal vertices. We call such a forest a *staircase forest*. If the forest is a tree, then we call it a *staircase tree*. Figure 2 shows the six staircase trees of size 4. Note that the children of any internal vertex v of a staircase forest have uniquely determined labels α or β, viz., the child to the right of v is labelled α, while the child below v is labelled β. We can label each root either α or β without changing the property of being a staircase tableau.

The first step in enumerating (α, β)-staircase tableaux by this forest approach is the case where the forest is a tree. Let $t(n)$ denote the number of staircase trees of size n. The root u must be in the upper-left corner (the $(1,1)$-entry). Let $v_1, \dots, v_n$ be the diagonal vertices, from top to bottom. The row subtree of u (i.e., the subtree whose root is the row child of u) can have any nonempty subset S of the diagonal vertices as endpoints, except for the conditions $v_1 \in S$ and $v_n \notin S$. If the row subtree has i endpoints, then there are $\binom{n-2}{i-1}$ ways to choose them, and then $t(i)$ ways to choose the subtree itself. Similarly there are $t(n-i)$ choices for the column subtree. Hence

$$t(n) = \sum_{i=1}^{n-1} \binom{n-2}{i-1} t(i)t(n-i), \tag{4.10}$$

with the initial condition $t(1) = 1$. The solution to this recurrence is clearly $t(n) = (n-1)!$, since the above sum will then have $n-1$ terms, all equal to $(n-2)!$.

The formula $t(n) = (n-1)!$ shows that $t(n)$ is equal to the number of n-cycles in the symmetric group $\mathfrak{S}_n$, while equation (4.10) shows that the number of staircase trees of size n whose row subtree has i endpoints, $1 \le i \le n-1$, is $(n-2)!$, independent of i. From this observation it is straightforward to give a bijection between staircase trees and n-cycles.

Alternatively, one can give an explicit bijection from staircase trees to cycles as follows. If v_i is incident to a vertical edge, then travel north from v_i as far as possible along edges of the staircase tree, then take a "zig-zag" path east-south, traveling on edges of the tree east and south and turning at each new vertex. This path will terminate at some vertex v_j where $j < i$; set $\pi(i) = j$. Similarly, if v_i is incident to a horizontal edge, then travel west from i as far as possible along edges of the tree, then take a zig-zag path south-east, traveling on edges of the tree south and east, and turning at each new vertex. This path will terminate at some v_j for $j > i$; set $\pi(i) = j$. The result will be a permutation π which is a single cycle. Using this bijection, the cycles associated to the first row of Figure 2 are, from left to right, (1234), (1324), and (1342), and the cycles associated to the second row are (1243), (1423), and (1432).

Let us now consider staircase forests F. We obtain such a forest by choosing a partition $\{B_1, \dots, B_k\}$ of the endpoints and then for each block B_i choosing a staircase tree whose endpoints are B_i. Since a staircase tree with endpoints B_i is equivalent to a cycle on the elements of B_i, we are just choosing a permutation of the endpoints. Hence there are $n!$ staircase forests of size n.

With little extra difficulty we can handle the labels α, β. Half the non-root vertices are row children, while half are column children. If F is a staircase forest and T a component of F with k endpoints, then T has $k-1$ row children, all labelled α, and $k-1$ column children, all labelled β. The root is labelled either α or β. Identifying the components of a staircase forest with the cycles of a permutation shows that

$$\sum_{\mathcal{T}} \mathrm{wt}(\mathcal{T}) = \sum_{w \in \mathfrak{S}_n} \mathrm{wt}(w),$$

where the first sum ranges over all (α, β)-staircase tableaux of size n, while in the second sum we define

$$\mathrm{wt}(w) = \prod_C (\alpha+\beta)(\alpha\beta)^{\#C-1},$$

where C ranges over all cycles of w. For instance, if $w = (1,3,6)(2,8)(4,9,7)(5)$ (disjoint cycle notation), then

$$\mathrm{wt}(w) = (\alpha+\beta)^4(\alpha\beta)^5.$$

A basic enumerative result on cycles of permutations states that if $\kappa(w)$ denotes the number of cycles of w then

$$F_n(x) := \sum_{w \in \mathfrak{S}_n} x^{\kappa(w)} = x(x+1)\cdots(x+n-1). \tag{4.11}$$

Hence

$$\begin{aligned}
\sum_{\mathcal{T}} \mathrm{wt}(\mathcal{T}) &= \sum_{w \in \mathfrak{S}_n} (\alpha+\beta)^{\kappa(w)}(\alpha\beta)^{n-\kappa(w)} \\
&= (\alpha\beta)^n F_n((\alpha+\beta)/\alpha\beta) \\
&= (\alpha+\beta)(\alpha+\beta+\alpha\beta)(\alpha+\beta+2\alpha\beta)\cdots(\alpha+\beta+(n-1)\alpha\beta). \qquad (4.12)
\end{aligned}$$

Equation (4.11) has (at least) two bijective proofs [27, Prop. 1.3.4], so these two proofs carry over to bijective proofs of equation (4.12).

Note that Theorem 4.1 implies that there are $4^n n!$ staircase tableaux of size n. One can give a simple bijection Φ from staircase tableaux to *doubly signed permutations* – that is, permutations where each position is decorated by two signs. The underlying permutation associated to a staircase tableaux is just the permutation associated to its staircase forest (after replacing γ's and δ's with α's and β's, respectively). The first sign associated to position i is $+$ if the ith diagonal box contains an α or δ, and is $-$, otherwise. The second sign which we associate to position i depends on the ith diagonal vertex and either the topmost vertex in the ith column or the leftmost vertex in the ith row. More specifically, if the ith diagonal vertex is α or γ and the leftmost vertex of the ith row is α or δ, we assign a $+$. If the ith diagonal vertex is α or γ and the leftmost vertex of the ith row is β or γ, we assign a $-$. On the other hand, if the ith diagonal vertex is β or δ and the topmost vertex of the ith column is α or β we assign a $+$. If the ith diagonal vertex is β or δ and the topmost vertex of the ith column is γ or δ we assign a $-$.

Remark 4.3. *Philippe Nadeau exhibits a simple recursive structure for alternative tableaux (staircase tableaux with no γ and δ)* [23]. *The results of this section can be derived with Nadeau's techniques. For example, Proposition 3.5 of* [23] *implies that there exist $4^n n!$ staircase tableaux of size n. James Merryfield also gave a bijective proof that the staircase tableaux have cardinality $4^n n!$* [22].

As a slight variant of equation (4.11), consider the problem of counting the number $g(n)$ of (α, β, γ)-staircase tableaux of size n. Substituting $\alpha+\gamma$ for α and setting $\alpha = \beta = \gamma = 1$ (or just setting $\alpha = 2$ and $\beta = 1$ in (4.11)) gives

$$g(n) = 3 \cdot 5 \cdots (2n+1) = (2n+1)!!,$$

the number of complete matchings on a $(2n+2)$-element set. By the interpretation in terms of cycles, we are counting permutations in $\mathfrak{S}_n$ where the least element in each cycle in 3-colored and the remaining elements are 2-colored. The third proof of [27, Prop. 1.3.4] gives a bijective proof of this result by first making three choices, then five choices, up to $(2n+1)$ choices. It is easy to encode these choices by a complete matching on $[2n+2]$, thereby giving a bijection between (α, β, γ)-staircase tableaux and matchings.

4.3. **Enumeration of staircase tableaux of a given type.** Recall from equation (1.1) that $Z_\sigma(\alpha, \beta, \gamma, \delta; q)$ is the generating polynomial for the staircase tableaux of type σ; here σ is a word in $\{\bullet, \circ\}^n$. By Theorem 1.5, the steady state probability that the ASEP is at state σ is proportional to $Z_\sigma(\alpha, \beta, \gamma, \delta; q)$. Therefore it is desirable to have explicit formulas for $Z_\sigma(\alpha, \beta, \gamma, \delta; q)$.

We do not have an explicit formula for $Z_\sigma(\alpha, \beta, \gamma, \delta; q)$ which works for arbitrary values of the parameters. However, a few special cases are known. In particular, we will discuss the following:

- an explicit formula for $Z_\sigma(1, 1, 1, 1; 1)$;
- an explicit formula for $Z_\sigma(1, 1, 0, 0; q)$;
- relations satisfied by $Z_\sigma(\alpha, \beta, \gamma, \delta; q)$;
- a recurrence for $Z_\sigma(\alpha, \beta, \gamma, 0; q)$.

4.3.1. *An explicit formula for* $Z_\sigma(1,1,1,1;1)$.

Proposition 4.4. *For any word* σ *in* $\{\bullet,\circ\}^n$, $Z_\sigma(1,1,1,1;1)=2^n n!$. *In other words, there are* $2^n n!$ *staircase tableaux of each type.*

Proof. This follows directly from the definition of staircase tableaux. Fix an arbitrary word σ in $\{\bullet,\circ\}^n$. Let us show that the staircase tableaux of type σ are in bijection with the staircase tableaux of type $\bullet^n$. Given a tableau T of type σ, we map it to a tableau $g(T)$ of type $\bullet^n$, by replacing every diagonal box which contains a β with a δ, and by replacing every diagonal box which contains a γ with an α. This is clearly a bijection. Since there are 2^n words of length n in $\{\bullet,\circ\}^n$, and the total number of staircase tableaux is $4^n n!$, there must be $2^n n!$ staircase tableaux of each type in $\{\bullet,\circ\}^n$. □

4.3.2. *An explicit formula for* $Z_\sigma(1,1,0,0;q)$. There is also an explicit formula when $\gamma=\delta=0$, and $\alpha=\beta=1$, which was found in [24] (though stated in terms of permutation tableaux). We first need to give a few definitions.

A *composition* of n is a list of positive integers which sum to n. If $I=(i_1,\dots,i_r)$ is a composition, let $\ell(I)=r$ be its number of parts. The *descent set* of I is $\mathrm{Des}(I)=\{i_1,\ i_1+i_2,\dots,i_1+\cdots+i_{r-1}\}$. We say that a composition J is *weakly coarser* than I, denoted $J\preceq I$, if J is obtained from I by merging some parts of I. For example, the compositions which are (weakly) coarser than the composition $(3,4,1)$ are $(3,4,1)$, $(7,1)$, $(3,5)$, and (8).

Given a word σ in $\{\bullet,\circ\}^n$, we associate to it a composition $I(\sigma)$ as follows. Read σ from right to left, and list the lengths of the consecutive blocks of $\circ$'s, between the right end of σ and the rightmost $\bullet$, between two $\bullet$'s, and between the leftmost $\bullet$ and the left end of σ. This gives $I'(\sigma)$. For example, if $\sigma=\bullet\circ\circ\circ\bullet\circ\circ$ then $I'(\sigma)=(2,3,0)$. Then we define $I(\sigma)$ by adding 1 to each entry of $I'(\sigma)$. So in this case, $I(\sigma)=(3,4,1)$.

We also define a relative of the q-factorial function: we define QFact as a function of any composition by

$$\mathrm{QFact}(j_1,\dots,j_p):=[p]_q^{j_1}[p-1]_q^{j_2}\dots[2]_q^{j_{p-1}}[1]_q^{j_p}.$$

Here $[p]_q:=1+q+\cdots+q^{p-1}$.

Finally, if $I\succeq J$, we define the statistic $st(I,J)$ by

$$st(I,J):=\#\{(i,j)\in\mathrm{Des}(I)\times\mathrm{Des}(J)|i\le j\}.$$

Theorem 4.5. [24, Theorem 4.2] *Let* σ *be any word in* $\{\circ,\bullet\}^n$, *and let* $I:=I(\sigma)$ *be the composition associated to* σ. *Then*

$$Z_\sigma(1,1,0,0;q)=\sum_{J\preceq I}(-1/q)^{l(I)-l(J)}q^{-st(I,J)}\mathrm{QFact}(J).$$

Example 4.6. *When* $\sigma=\bullet\circ\circ\circ\bullet\circ\circ$, *we have* $I:=I(\sigma)=(3,4,1)$. *The compositions coarser than* I *are* $(3,4,1)$, $(7,1)$, $(3,5)$, *and* (8), *so we get*

$$\begin{aligned}Z_\sigma(1,1,0,0;q)&=q^0q^{-3}[3]_q^3[2]_q^4[1]_q^1-q^{-1}q^{-2}[2]_q^7[1]_q^1-q^{-1}q^{-1}[2]_q^3[1]_q^5+q^{-2}q^0[1]_q^8\\&=q^7+7q^6+24q^5+52q^4+76q^3+75q^2+47q+15.\end{aligned}$$

This is the generating polynomial for staircase tableaux of type $\bullet\circ\circ\circ\bullet\circ\circ$ *when* $\alpha=\beta=1$ *and* $\gamma=\delta=0$.

4.3.3. *Relations satisfied by $Z_\sigma(\alpha,\beta,\gamma,\delta;q)$.* In this subsection we will use Z_σ as shorthand for $Z_\sigma(\alpha,\beta,\gamma,\delta;q)$. We will recall here some relations satisfied by Z_σ that were proved in [11, 12]. Given any word σ in the alphabet $\{\bullet,\circ\}$, let $\ell(\sigma)$ denote the length of σ.

Theorem 4.7. [11, 12] *Let $\alpha,\beta,\gamma,\delta,q$ be arbitrary parameters, and let λ_n be defined by $\lambda_n = \alpha\beta - \gamma\delta q^{n-1}$ for $n \geq 1$. Let σ_1,σ_2,σ be arbitrary words in the alphabet $\{\bullet,\circ\}$. Then we have the following relations among the $Z_\sigma = Z_\sigma(\alpha,\beta,\gamma,\delta;q)$.*

$$Z_{\sigma_1\bullet\circ\sigma_2} - qZ_{\sigma_1\circ\bullet\sigma_2} = \lambda_{\ell(\sigma_1)+\ell(\sigma_2)+2}(Z_{\sigma_1\bullet\sigma_2} + Z_{\sigma_1\circ\sigma_2}). \tag{4.13}$$

$$\alpha Z_{\circ\sigma} - \gamma Z_{\bullet\sigma} = \lambda_{\ell(\sigma)+1} Z_\sigma. \tag{4.14}$$

$$\beta Z_{\sigma\bullet} - \delta Z_{\sigma\circ} = \lambda_{\ell(\sigma)+1} Z_\sigma. \tag{4.15}$$

Note that the proof of Theorem 4.7 in [11, 12] used a complicated induction and was not very combinatorial.

4.3.4. *A recurrence for $Z_\sigma(\alpha,\beta,\gamma,0;q)$.* However, when $\delta = 0$, Theorem 4.7 simplifies, and we can give a purely combinatorial proof using staircase tableaux. Throughout this section our staircase tableaux will be assumed to have no δ's, and we will abbreviate $Z_\sigma(\alpha,\beta,\gamma,0;q)$ by Z_σ.

Theorem 4.8. *Let σ,σ_1,σ_2 be arbitrary words in the alphabet $\{\bullet,\circ\}$. Then we have the following.*

$$Z_{\sigma_1\bullet\circ\sigma_2} = qZ_{\sigma_1\circ\bullet\sigma_2} + \alpha\beta(Z_{\sigma_1\bullet\sigma_2} + Z_{\sigma_1\circ\sigma_2}), \tag{4.16}$$

$$\alpha Z_{\circ\sigma} = \gamma Z_{\bullet\sigma} + \alpha\beta Z_\sigma, \tag{4.17}$$

$$Z_{\sigma\bullet} = \alpha Z_\sigma. \tag{4.18}$$

Proof. This recurrence is best explained with pictures. We begin with equations (4.18) and (4.17), which are easiest to prove. To prove (4.18), it suffices to note that any staircase tableau (with no δ's) whose type ends with $\bullet$ must have an α in its lower left square. The weight of such a tableau is α times the weight of the tableau obtained from it by deleting the leftmost column. See the left part of Figure 3. Equation (4.18) follows.

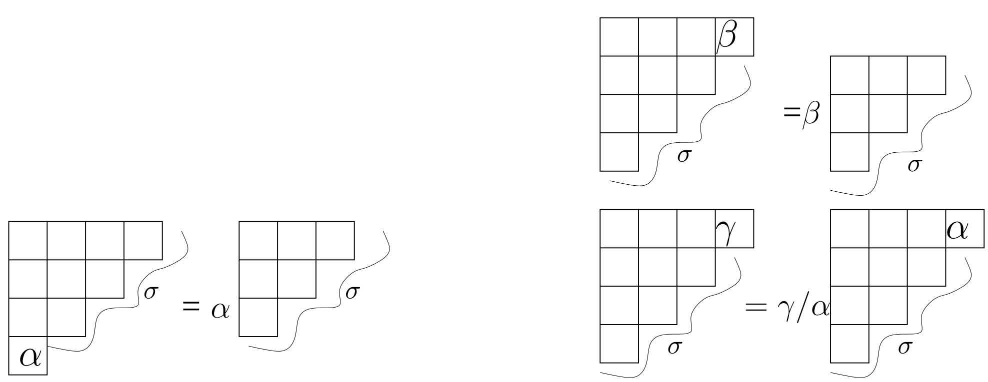

FIGURE 3. The left and right parts of the picture prove equations (4.18) and (4.17)

To prove (4.17) note that any staircase tableau whose type begins with $\circ$ must have a β or a γ in its upper right square. If it has a β there, then the weight of that tableau is equal to β times the weight of the tableau obtained by deleting the topmost row. Alternatively,

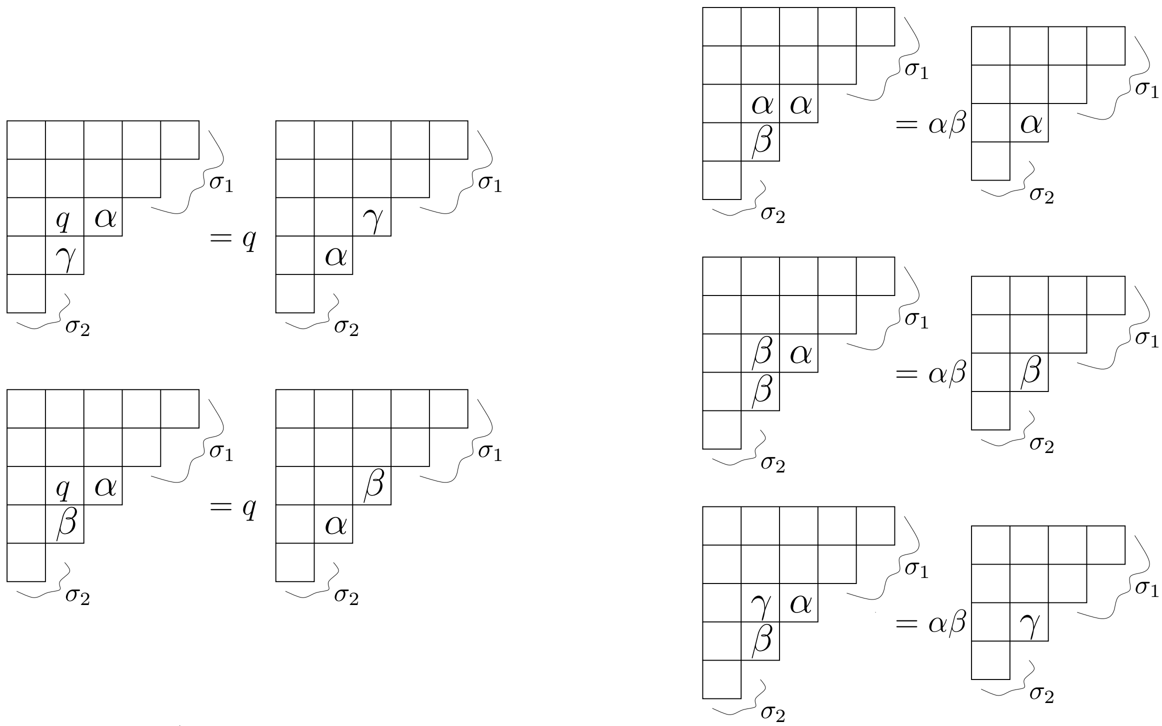

FIGURE 4. This picture proves equation (4.16).

if it has a γ there, then its weight is equal to $\frac{\gamma}{\alpha}$ times the weight of the tableau obtained from it by replacing the γ with an α. See the left right of Figure 3.

To prove (4.16), note that if a staircase tableau has type $\sigma_1 \bullet \circ \sigma_2$, then its two diagonal boxes corresponding to the $\bullet\circ$ must be either $\alpha\gamma$ or $\alpha\beta$. If the two boxes are $\alpha\gamma$, then the box above the γ and left of the α will get filled with q. If the two boxes are $\alpha\beta$, then the box above the β and left of the α may be filled with either a q, α, β or γ. These five possibilities are shown in Figure 4.

As shown in the left of Figure 4, if that third box is a q, then the generating polynomial for such staircase tableaux of type $\sigma_1 \bullet \circ \sigma_2$ is equal to q times the generating polynomial for staircase tableau of type $\sigma_1 \circ \bullet \sigma_2$. One can prove this bijectively by taking the two columns above the q and α and swapping them; and by taking the two rows left of the q and γ (respectively q and β) and swapping them. (The box filled with the q will become a box filled with $u = 1$.)

On the other hand, if the two diagonal boxes are α and β, and the box above the β and left of the α is filled with either α, β, or γ, then the weight of this tableau is equal to $\alpha\beta$ times the weight of the tableau obtained by deleting the column with the α in the diagonal box, and deleting the row with the β in the diagonal box. This completes the proof of (4.16). □

4.4. **More factorizations of the partition function.** Theorem 4.7 is very useful for proving various factorizations of the partition function.

Proposition 4.9.

$$Z_n(1;\alpha,\beta,\gamma,-\beta;q) = \prod_{j=0}^{n-1}(\alpha + q^j\gamma).$$

Proof. We use Theorem 4.7, with $\delta = -\beta$. In this case we have $\lambda_n = \beta(\alpha + q^n\gamma)$, and so $Z_{\sigma\bullet} + Z_{\sigma\circ} = (\alpha + q^{\ell(\sigma)+1}\gamma)Z_\sigma$. We now use induction on n. Note that $Z_n = \sum_\sigma Z_\sigma$, where the sum is over all words $\sigma \in \{\bullet, \circ\}^n$. Then $Z_{n+1} = \sum_\sigma (Z_{\sigma\bullet} + Z_{\sigma\circ}) = \sum_\sigma (\alpha + q^n\gamma)Z_\sigma = (\alpha + q^n\gamma)Z_n$. The results now follows by induction. □

Proposition 4.10.

$$Z_n(-1;\alpha,\beta,\gamma,\beta;q) = (-1)^n\prod_{j=0}^{n-1}(\alpha - q^j\gamma).$$

Proof. Exercise. The proof is analogous to the proof of Proposition 4.9. □

Proposition 4.11.

$$Z_n(y;\alpha,\alpha,\alpha,\alpha;-1) = 0 \text{ for } n \geq 3.$$

Proof. We use Theorem 1.13. Our choice of specialization makes the sum over j equal to 0 unless $k = 0$ or $k = 1$. (To see this, split the sum over j into sums over even and odd j, and consider even and odd k.) The k-sum has the term $(q^k;q)_{n-k}$, which is zero for $q = -1$ when either $k = 0$ and $n > 0$, or $k = 1$ and $n > 2$. So if $n > 2$ we get 0. □

Remark 4.12. *Note that one can generalize the preceeding proposition and prove that* $Z_n(y;\alpha,\alpha,\alpha,\alpha;q) = 0$ *for* $n > 2m$ *and* $q^m = -1$.

4.5. **Enumeration of staircase tableaux when $\beta = 1$ and $\delta = 0$.** As we've seen in Section 4.3.4, the combinatorics of staircase tableaux becomes a bit simpler when $\delta = 0$. In this section we explore the combinatorics when in addition we impose $q = 1$. By Theorem 4.1, the generating polynomial for staircase tableaux with no δ's is

$$\prod_{j=0}^{n-1}(\alpha + \beta + \gamma + j\beta(\alpha + \gamma)).$$

Therefore the number of staircase tableaux of size with no δ is $(2n+1)!! = (2n+1)\cdot(2n-1)\cdot\ldots\cdot 3\cdot 1$. Since $(2n+1)!!$ is the number of perfect matchings of the set $\{1,2,\ldots,2n+2\}$, this implies the following.

Corollary 4.13. *There exists a bijection between the staircase tableaux of size n with no δ and the perfect matchings of* $\{1,2,\ldots,2n+2\}$.

We gave the sketch of a bijective proof of this result in Section 4.2. Now let us study the combinatorics of those tableaux with no δ in the case $\beta = 1$. Let $Z_n(\alpha,\gamma;q) = Z_n(1;\alpha,1,\gamma,0;q)$.

Proposition 4.14. *The generating polynomial $Z_n(\alpha,\gamma;q)$ of staircase tableaux of size n is equal to the generating polynomial of weighted Dyck paths of length $2n+2$ where the North-East steps get weight 1 and the South East steps starting at height i have weight*

- $(\alpha + \gamma q^i)[i+1]_q$ *if* $k = 2i+2$
- $q^i + (\alpha + \gamma q^i)[i]_q$ *if* $k = 2i+1$.

Proof. From Corollary 2.10, we know that Z_n with $\delta = 0$ is a factor times the moments of the orthogonal polynomials with

$$b_n = \frac{2 + q^n((a+b+c) + (1-q^{n-1}(1+q))abc)}{1-q}$$

and

$$\lambda_n = \frac{(1-q^n)(1-q^{n-1}ab)(1-q^{n-1}ac)(1-q^{n-1}bc)}{(1-q)^2}.$$

If $\beta = 1$, then $b = -q$. We get that Z_n is exactly equal to the moments of the polynomials with

$$b_n = [n+1]_q(\alpha + \gamma q^n) + (\alpha + \gamma q^n)[n]_q + q^n, \quad \text{and } \lambda_n = [n]_q(\alpha + \gamma q^{n-1})((\alpha + \gamma q^n)[n]_q + q^n).$$

Now we use a result on page 46 of [5] which says that if $G_n(x)$ are orthogonal polynomials with b_n and λ_n arbitrary, then $H_{2n+1}(x) = xG_n(x^2)$ are orthogonal polynomials with $B_n = 0$ and $\Lambda_1 = 1$, $\Lambda_{2n+2} = b_n - \Lambda_{2n+1}$ and $\Lambda_{2n+1} = \lambda_n/\Lambda_{2n}$. We get that he generating polynomial of staircase tableaux of size n $Z_n(\alpha, \gamma; q)$ is equal to the moments μ_{2n+2} of the orthogonals polynomials defined by

$$xH_n(x) = H_{n+1}(x) + \Lambda_n H_{n-1}(x)$$

with $\Lambda_{2n} = (\alpha + \gamma q^{n-1})[n]_q$ and $\Lambda_{2n+1} = (\alpha + \gamma q^n)[n]_q + q^n$. As Dyck paths are Motzkin paths with no east steps, the proposition follows using Theorem 2.2. $\square$

Remark. When $q = \alpha = \gamma = 1$, it is well known that these paths are in bijection with perfect matchings of $\{1, 2, \ldots, 2n+2\}$ [17]. If the South East steps starting at height i had weight $[i]_q$, these paths would correspond to moments of the classical q-Hermite polynomials or perfect matchings counted by crossings (see [25] and references wherein).

We can now give a combinatorial interpretation of the preceeding proposition. A matching of $\{1, \ldots, 2n+2\}$ is a sequence of $n+1$ mutually disjoint edges (i, j) with $1 \le i < j \le 2n+2$. Given an edge $e = (i, j)$, let $cross(e)$ be the number of edges (ℓ, k) such that $i < \ell < j < k$ and $nest(e)$ be the number of edges (ℓ, k) such that $\ell < i < j < k$. We define the f-crossing of an edge e to be equal to $cross(e)$ if $nest(e) > 0$ and $\lfloor cross(e)/2 \rfloor$ otherwise. We said that an edge is nested (resp. crossed) if $cross(e) < nest(e)$ (resp. if $cross(e) > nest(e)$).

Theorem 4.15. *There exists a bijection between staircase tableaux of size n with j entries equal to q, k entries equal to α and ℓ entries equal to γ and matchings of $\{1, \ldots, 2n+2\}$ where j is the number of f-crossings, k is the number of nested edges and ℓ the number of crossed edges.*

Proof. The proof is direct using the classical bijection between labelled Dyck paths and matchings. See [25] for example. $\square$

5. Open problems

We conclude this paper with a list of open problems.

Problem 5.1. *Give combinatorial proofs of Propositions 4.9, 4.10, and 4.11, using appropriate involutions on staircase tableaux.*

Problem 5.2. *Recall from equation (1.1) that $Z_\sigma(\alpha,\beta,\gamma,\delta;q)$ is the generating polynomial for the staircase tableaux of type σ; here σ is a word in $\{\bullet,\circ\}^n$. By Theorem 1.5, the steady state probability that the ASEP is at state σ is proportional to $Z_\sigma(\alpha,\beta,\gamma,\delta;q)$. Find an explicit formula for $Z_\sigma(\alpha,\beta,\gamma,\delta;q)$.*

Problem 5.2 is probably quite difficult. Problem 5.3 should be more tractable, however, since one has a simple recurrence for $Z_\sigma(\alpha,\beta,\gamma,0;q)$ given by Theorem 4.8.

Problem 5.3. *Find an explicit formula for $Z_\sigma(\alpha,\beta,\gamma,0;q)$.*

Problem 5.4. *Recall the definition of the bijection Φ from Secton 4.2. Find a statistic $s(\pi)$ on doubly signed permutations, which corresponds to the q statistic on staircase tableaux via Φ. More specifically, we require that q^r is the maximal power of q dividing $\mathrm{wt}(\mathcal{T})$ if and only if $s(\Phi(\mathcal{T}))=r$.*

Problem 5.5. *If we restrict Φ to the set of staircase tableaux of size n of a given type, then for all i, the first sign associated to position i is the same for all tableaux. Therefore if we forget the first sign, we get a bijection from the $2^n n!$ staircase tableaux of a given type to signed permutations. For any fixed type σ, can one find a statistic $s_\sigma(\pi)$ on signed permutations which corresponds to the q statistic on the staircase tableaux of type σ?*

Problem 5.6. *Find an explicit formula for $Z_n(y;\alpha,\beta,\gamma,\delta;q)$ from which it is obvious that $Z_n(y;\alpha,\beta,\gamma,\delta;q)$ is a polynomial with positive coefficients. Such a formula can be found when $\gamma=\delta=0$ [20].*

Problem 5.7. *Prove that $Z_n(y;\alpha,\beta,\gamma,\delta;q)$ is equal to $y^n Z_n(1/y;\beta,\alpha,\delta,\gamma;q)$ by exhibiting an involution on staircase tableaux.*

Problem 5.8. *Give a simple bijection proving that the numbers $Z_n(y;\alpha,\beta,\gamma,\delta;q)$ are given by Sloane's sequence A026671 (enumerating certain lattice paths) when $\alpha=\beta=\gamma=y=1$ and $\delta=q=0$. Note that Corollary 3.10 gives a (non-bijective) proof of this equality, by showing that the generating functions of both sets of numbers are equal. See also Example 3.11.*

Problem 5.9. *Can one find other formulas for the moments of Askey Wilson polynomials? In particular, is there a formula that makes manifest the symmetry in a,b,c,d?*

This is possible when at least one of a,b,c,d is 0. In particular, Josuat-Vergès [20] gave a strictly polynomial version of the Askey Wilson moments when $c=d=0$.

$$\begin{aligned} 2^n\mu_n(a,b) &= \sum_{t=0}^{n}\sum_{p=0}^{\lfloor (n-t)/2\rfloor}\sum_{i=0}^{p}\left(\binom{n}{p-i}-\binom{n}{p-i-1}\right)(-1)^i q^{\binom{i+1}{2}} \\ &\quad\times \begin{bmatrix} n-2p \\ t \end{bmatrix}_q \begin{bmatrix} n-2p+i \\ i \end{bmatrix}_q b^t a^{n-t-2p}. \end{aligned}$$

This can be proved from our Theorem 1.12 with $c=d=0$, using the q-binomial theorem, the binomial theorem and the terminating ${}_2\phi_1(x)$ at $x=q$ [18].

Using the same techniques, one can obtain a formula for $d=0$,

$$\begin{aligned} 2^n \mu_n(a,b,c) &= \sum_{t_1=0}^{n} \sum_{t_2=0}^{n} \sum_{p=0}^{\min[\lfloor (n-t_1)/2 \rfloor, \lfloor (n-t_2)/2 \rfloor]} \sum_{F=\max[p,\lfloor (n-t_1-t_2)/2 \rfloor]}^{min[n-p-t_1, n-p-t_2]} \left(\binom{n}{p} - \binom{n}{p-1} \right) \\ &\times (-1)^{n-t_1-t_2-F-p} q^{\binom{n-t_1-t_2-F-p+1}{2}} \\ &\times \begin{bmatrix} t_1+F-p \\ t_1 \end{bmatrix}_q \begin{bmatrix} t_2+F-p \\ n-F-p-t_1 \end{bmatrix}_q \begin{bmatrix} n-F-p \\ t_2 \end{bmatrix}_q b^{t_1} c^{t_2} a^{-n+t_1+t_2+2F}. \end{aligned}$$

This is obviously symmetric in a, b and $c=0$. One can get back the previous equation by setting $c=0$, so $t_2=0$, and then put $F=n-t_1-p$.

Problem 5.10. *Find a combinatorial proof of the formula for $\mu_n(a,b,c)$ above.*

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